Characterization of Simplicity and Cancellativity in $\beta S$

Neil Hindman

and

Dona Strauss

Abstract. We determine precisely when the Stone-Čech compactification $\beta S$ of a discrete semigroup $S$ is simple and when it is left cancellative or right cancellative. As a consequence we see that $\beta S$ is cancellative only when it is trivially so. That is, $\beta S$ is cancellative if and only if $S$ is a finite group.

Given a discrete semigroup $S$, the operation can be extended to the Stone-Čech compactification $\beta S$ of $S$ so that $\beta S$ is a right topological semigroup with $S$ contained in its topological center. (That is, given any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = qp$ is continuous and, given any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = xq$ is continuous.) This implies that, for any $x, y \in \beta S$, $xy = \lim_{s \to x} \lim_{t \to y} st$ where $s$ and $t$ denote elements of $S$.

It has been known for a long time that left or right cancellativity is rare in $\beta S$ for a typical semigroup $S$ and several papers have been published on the subject. See [3, Chapter 8] and the notes to that chapter for a summary of what was known in 1998. To the best of our knowledge, there have not been more recent results.

A subset $T$ of a semigroup $S$ is called a left ideal if it is non-empty and if $ST \subseteq T$; it is called a right ideal if it is non-empty and if $TS \subseteq T$. It is called an ideal if it is both a left ideal and a right ideal. $S$ is said to be simple if it has no proper subsets which are ideals.

In this note we characterize discrete semigroups $S$ for which $\beta S$ is simple and those for which $\beta S$ is left cancellative or right cancellative. We regard $\beta S$ as being the set of ultrafilters on $S$, with the points of $S$ identified with the principal ultrafilters. The topology of $\beta S$ is defined by choosing the sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$ as a base, where $A$ denotes a subset of $S$. With this topology, $\overline{A} = \text{cl}_{\beta S}(A)$ and is clopen in $\beta S$.

Our characterizations of cancellativity heavily involve the smallest ideal of $\beta S$. If a semigroup $S$ has a minimal left ideal or a minimal right ideal which contains an

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idempotent, then it has a smallest two sided ideal $K(S)$. $K(S)$ is the union of all minimal left ideals of $S$ and is the union of all minimal right ideals of $S$. Given any minimal left ideal $L$ and any minimal right ideal $R$ of $S$, $L \cap R$ is a group and all groups of this form are isomorphic. The group $L \cap R$ is called the structure group of $S$. Thus $K(S)$ is the disjoint union of groups, each isomorphic to the structure group.

Every compact right topological semigroup does have a minimal left ideal which contains an indempotent. Thus, if $S$ is an arbitrary discrete semigroup, the preceding remarks hold for $\beta S$. See [1] or [3] for a derivation of these facts.

If $V$ is a subset of a semigroup $S$, $E(V)$ will denote the set of idempotents in $V$. We shall use the fact that, for any semigroup $S$, $E(L)$ is a left zero semigroup if $L$ is a minimal left ideal of $S$ which contains an idempotent [3, Lemma 1.30(b)]. Dually, $E(R)$ is a right zero semigroup if $R$ is a minimal right ideal of $S$ which contains an idempotent.

We need the following simple lemmas. When we say that two semigroups with topologies are topologically isomorphic we mean that there is a bijection between them which is simultaneously an isomorphism and a homeomorphism.

1 Lemma. Let $S$ and $T$ be discrete semigroups with $S$ being finite. Then $\beta(S \times T)$ is topologically isomorphic to $S \times \beta T$. If $T$ is a right zero semigroup, so is $\beta T$; if $T$ is a left zero semigroup, so is $\beta T$.

Proof. Let $\pi_1$ and $\pi_2$ denote the projection maps of $S \times T$, and let $\bar{\pi}_1 : \beta(S \times T) \to S$ and $\bar{\pi}_2 : \beta(S \times T) \to \beta T$ denote their continuous extensions. Define $\theta : \beta(S \times T) \to S \times \beta T$ by $\theta(x) = (\bar{\pi}_1(x), \bar{\pi}_2(x))$.

Since $\theta$ is the continuous extension of the inclusion map of $S \times T$ in $S \times \beta T$, which is a homomorphism, $\theta$ is a homomorphism [3, Corollary 4.22]. Since $S \times T$ is dense in $S \times \beta T$, $\theta$ is surjective. To see that $\theta$ is injective, note that, for each $s \in S$, $\pi_2$ is injective on $\{s\} \times T$. So $\bar{\pi}_2$ is injective on $c\ell_{\beta(S \times T)}(\{s\} \times T)$ [3, Exercise 3.4.1]. Let $x$ and $y$ be distinct elements of $\beta(S \times T)$. If $\bar{\pi}_1(x) = \bar{\pi}_1(y) = s$, then $x, y \in c\ell_{\beta(S \times T)}(\{s\} \times T)$ and so $\bar{\pi}_2(x) \neq \bar{\pi}_2(y)$. Thus $\theta(x) \neq \theta(y)$.

If $T$ is a right zero semigroup then, for every $x_1, x_2 \in \beta T$,

$$x_1 x_2 = \lim_{t_1 \to x_1} \lim_{t_2 \to x_2} t_1 t_2 = \lim_{t_2 \to x_2} t_2 = x_2.$$

Thus $\beta T$ is a right zero semigroup. Similarly, if $T$ is a left zero semigroup, $\beta T$ is also a left zero semigroup. \hfill \Box
2 Lemma. Let $S$ be a discrete semigroup and let $G$ be a compact subgroup of $\beta S$. Then $G$ is finite.

Proof. By [2, Theorem 14.25], $\beta S$ is an $F$-space, and therefore $G$ is an $F$-space. Let $e$ be the identity of $G$. Given any $g, h \in G$, $\rho_{h^{-1}g}$ is a homeomorphism from $G$ to $G$ taking $h$ to $g$. That is, $G$ is homogeneous. But by [4, Corollary 3.4.2], no infinite compact $F$-space is homogeneous. □

3 Lemma. Let $S$ be a discrete semigroup. If $K(\beta S) \cap S \neq \emptyset$, then there is an idempotent $e \in K(\beta S) \cap S$, $G = (e\beta S) \cap (\beta Se)$ is a finite group, and $G \subseteq S$. Furthermore, $Se$ is a minimal left ideal of $S$.

Proof. Pick $x \in K(\beta S) \cap S$ and pick a minimal left ideal $L$ of $\beta S$ and a minimal right ideal $R$ of $\beta S$ such that $x \in L \cap R$. Then $G = L \cap R$ is a group. Since $L = \beta Sx = \rho_{x}[\beta S]$ and $R = x\beta S = \lambda_{x}[\beta S]$ we have that $G$ is compact, hence finite by Lemma 2. The powers of $x$ are then a finite group, so the identity $e$ of $G$ is in $S$. To see that $G \subseteq S$, let $g \in G$. Let $g^{-1}$ be the inverse of $g$ in $G$. Then $e = \rho_{g^{-1}}(g)$ and $\{e\}$ is a neighborhood of $e$ so pick a neighborhood $V$ of $g$ such that $\rho_{g^{-1}}[V] \subseteq \{e\}$. Since $S$ is dense in $\beta S$, pick $y \in V \cap S$. Then $yg^{-1} = e$ so $ye = eg = g$. Since $y \in S$ and $e \in S$, $g \in S$.

To see that $Se$ is a minimal left ideal of $S$, suppose that $T$ is a left ideal of $S$ for which $T \subseteq Se$. Then $c\ell_{\beta S}(T)$ is a left ideal of $\beta S$ contained in the minimal left ideal $\beta Se$ of $\beta S$ [3, Theorem 2.17]. So $c\ell_{\beta S}(T) = \beta Se \supseteq Se$. Since the points of $S$ are isolated in $\beta S$, it follows that $T \supseteq Se$. □

We can now characterize the semigroups $S$ for which $\beta S$ is simple.

4 Theorem. Let $S$ be any semigroup. The following statements are equivalent:

(a) $\beta S$ is simple.

(b) $S$ is a simple semigroup with a minimal left ideal containing an idempotent. Furthermore, the structure group of $S$ is finite and $S$ has only a finite number of minimal left ideals or only a finite number of minimal right ideals.

(c) $S$ contains a finite group $G$, a left zero semigroup $X$ and a right zero semigroup $Y$ such that $S$ is isomorphic to the semigroup $X \times G \times Y$ with the semigroup operation defined by $(x, g, y)(x', g', y') = (x, gxy'g', y')$ for every $x, x' \in X, g, g' \in G, y, y' \in Y$. Furthermore, either $X$ or $Y$ is finite.

Proof. (a) $\Rightarrow$ (b). By Lemma 3, $S$ has a minimal left ideal which contains an idempotent $e$. By [3, Theorem 1.65], $K(S) = S \cap K(\beta S) = S \cap \beta S = S$ and so $S$ is simple. Also,
by Lemma 3, $G = (e\beta S) \cap (\beta Se) = e\beta Se$ is a finite group. Given $x \in G$, $x \in S$ so $x = exe \in e\beta Se$ so $G = e\beta Se$. Since $e$ is a minimal idempotent, $G = e\beta Se$ is the structure group of $S$.

Now suppose that $S$ has an infinite number of minimal left ideals and an infinite number of minimal right ideals and choose sequences $(L_n)_{n=1}^{\infty}$ and $(R_n)_{n=1}^{\infty}$ of distinct minimal left and right ideals respectively. For each $n$, let $x_n$ be the identity of $Se \cap R_n$ and let $y_n$ be the identity of $eS \cap L_n$. Notice that for each $n$, $x_ny_n \in R_n \cap L_n$. In particular, $D = \{x_ny_n : n \in \mathbb{N}\}$ is infinite. Pick $q \in \overline{D} \setminus D$. Pick a minimal left ideal $I$ of $\beta S$ such that $q \in I$. Then $I = \beta Sq$ so $q \in \beta Sq = cl(Sq)$ so pick $s \in S$ such that $D \in sq$. Then $\{t \in S : st \in D\} \in q$, so pick $t \neq v$ in $D$ such that $st \in D$ and $sv \in D$. Pick $k,l,m,n \in \mathbb{N}$ such that $t = x_ky_k$, $v = x_ly_l$, $st = x_my_m$, and $sv = x_ny_n$. Now $st \in x_mS = R_m$ so $s \in R_m$. Also $sv \in x_nS = R_n$ so $s \in R_n$. Consequently $n = m$ and thus $st = sv \in L_n \cap R_n$. But $t \in L_k$ so $st \in L_k$ and $v \in L_l$ so $sv \in L_l$ so $k = l = n$ and thus $t = v$, a contradiction.

$(b) \Rightarrow (c)$. By [3, Theorem 1.64], there exist a group $G$, a left zero semigroup $X$ and a right zero semigroup $Y$ such that $K(S)$ is isomorphic to the semigroup $X \times G \times Y$ with the semigroup operation as in the statement of $(c)$. Since $S$ is simple, $S = K(S)$. Since the structure group of $X \times G \times Y$ is $G$, $G$ is finite. For each $x \in X$, $\{x\} \times G \times Y$ is a minimal right ideal of $X \times G \times Y$ and for each $y \in Y$, $X \times G \times \{y\}$ is a minimal left ideal of $X \times G \times Y$, so either $X$ or $Y$ is finite.

$(c) \Rightarrow (a)$. Assume that $X$ is finite. Notice that $X \times G \times \beta Y$ endowed with the semigroup operation $(x,g,\eta)(x',g',\eta') = (x,g\eta x'g',\eta')$ is a compact right topological semigroup with $X \times G \times Y$ contained in its topological center. For this first note that given $x' \in X$, $g,g' \in G$, and $\eta \in \beta Y$, one has

$$g\eta x'g' = \lambda_g \circ \rho_{x'g'}(\eta) \in \lambda_g \circ \rho_{x'g'}[cl(\beta S(Y))] = cl(\beta S(gYx'g')) \subseteq cl(\beta S(G)) = G.$$ 

It is then routine to verify that the operation on $X \times G \times \beta Y$ is associative and that $X \times G \times \beta Y$ is a right topological semigroup with $X \times G \times Y$ contained in its topological center.

We claim that $\beta S$ is isomorphic to $X \times G \times \beta Y$. By Lemma 1, $\beta S$ can be identified as a topological space with $X \times G \times \beta Y$. (We cannot invoke the algebraic portion of Lemma 1 because the operation on $X \times G \times Y$ is not the direct product.) The semigroup operation of $\beta S$ is characterized by

$$(x,g,\eta)(x',g',\eta') = \lim_{y \to \eta} \lim_{y' \to \eta'} (x,g,y)(x',g',y') = (x,g\eta x'g',\eta')$$
where the last equality holds because \( X \times G \times \beta Y \) is a right topological semigroup with \( X \times G \times Y \) contained in its topological center.

We now claim that \( X \times G \times \beta Y \) is simple. To see this, for each \( x \in X \), let \( R_x = \{(x, g, \eta) : g \in G \text{ and } \eta \in \beta Y\} \). We will show that each \( R_x \) is a minimal right ideal of \( X \times G \times \beta Y \). Since \( X \times G \times \beta Y = \bigcup_{x \in X} R_x \), this will suffice. So let \( x \in X \) and pick a minimal right ideal \( R \subseteq R_x \). Pick \( g \in G \) and \( \eta \in \beta Y \) such that \((x, g, \eta) \in R\). To see that \( R_x \subseteq R \), let \( g' \in G \) and \( \eta' \in \beta Y \) be given. Now \( g\eta xe \in G \) so pick \( h \in G \) such that \( g\eta xe = g\eta xe h = g' \). Then \((x, g', \eta') = (x, g, \eta)(x, h, \eta') \in R\).

An analogous argument shows that \( \beta S \) is simple if \( Y \) is finite.

We now characterize the semigroups \( S \) for which \( \beta S \) is left or right cancellative.

5 Theorem. Let \( S \) be a discrete semigroup. The following statements are equivalent.

(a) \( \beta S \) is left cancellative.
(b) \( K(\beta S) \) contains an element which is left cancelable in \( \beta S \).
(c) \( \beta S \) has a left cancelable element and \( K(\beta S) = \beta S \).
(d) Every idempotent in \( \beta S \) is a left identity for \( \beta S \).
(e) There exist a finite group \( G \subseteq S \) and a compact Hausdorff right zero semigroup \( T \subseteq \beta S \) such that \( \beta S = GT \) and the function \( \varphi : G \times T \to \beta S \) defined by \( \varphi(g, \eta) = g\eta \) is both an isomorphism and a homeomorphism.
(f) There exist a finite group \( G \) and a right zero semigroup \( R \) such that \( S \) is isomorphic to \( G \times R \).
(g) There exist a finite group \( G \) and a discrete right zero semigroup \( R \) such that \( \beta S \) is topologically isomorphic to \( G \times \beta R \).
(h) \( S \) has a left cancelable element and \( \beta S \) is simple.

Proof. (a) \( \Rightarrow \) (b). This is obvious.

(b) \( \Rightarrow \) (c). Suppose one has \( p \in \beta S \setminus K(\beta S) \). Pick a left cancelable \( q \in K(\beta S) \) and pick a minimal right ideal \( R \) of \( \beta S \) such that \( q \in R \). Then \( R = q\beta S = qR \). So \( qp = qr \) for some \( r \in R \subseteq K(\beta S) \), a contradiction.

(c) \( \Rightarrow \) (d). Suppose that \( x \in K(\beta S) \) is left cancelable in \( \beta S \). We can choose an idempotent \( p \in K(\beta S) \) for which \( xp = x \). Then \( xpy = xy \) and so \( py = y \) for every \( y \in \beta S \). This shows that \( \beta S = p\beta S \) and hence that \( \beta S \) is a minimal right ideal of \( \beta S \). It follows that \( \beta S = q\beta S \) for every idempotent \( q \in \beta S \). So, if \( y \in \beta S \), \( y = qz \) for some \( z \in \beta S \) and \( qy = qz = qz = y \).
(d) ⇒ (e). We have that $\beta S$ is simple, because $\beta S = p\beta S \subseteq K(\beta S)$ for any minimal idempotent $p$ in $\beta S$. It follows from Lemma 3 that $S$ contains a minimal idempotent $e$ of $\beta S$ for which $G = (e\beta S) \cap (\beta Se) = e\beta Se$ is a finite subgroup of $S$. Let $T = E(\beta S)$. Then each $\eta \in T$ is a left identity for $\beta S$ so $T = \bigcap_{y \in \beta S} \rho_y^{-1}\{y\}$ so $T$ is compact.

It is routine to verify that $\varphi$ is a continuous homomorphism. To see that $\varphi$ is surjective let $q \in \beta S$ and pick $p \in T$ such that $q \in \beta Sp$. Then $qp = p$ and $e\varphi e \in G$ so $q = \varphi(e\varphi e, p)$. To see that $\varphi$ is injective, suppose that $g, h \in G$ and $y, z \in T$. Then $g = ge = gye = hze = he = h$ so $gy = gz$. Then $y = ey = g^{-1}gy = g^{-1}gz = ez = z$.

(e) ⇒ (f). The assumption that $\beta S = GT$ implies that $e$ is a left identity for $\beta S$. Let $R = T \cap S$. Then $\varphi[G \times R] \subseteq S$. To see that $\varphi[G \times R] = S$ let $x \in S$. Pick $g \in G$ and $y \in T$ such that $x = gy$. Then $g^{-1}x = ey = y$ because $e$ is a left identity for $\beta S$.

(f) ⇒ (g). This follows from Lemma 1.

(g) ⇒ (a). Since $\beta R$ is a right zero semigroup, this is immediate.

At this stage we have shown that statements (a) through (g) are equivalent. By [3, Lemma 8.1] we have that (h) implies (c). Statements (a) and (c) trivially imply (h). □

6 Theorem. Let $S$ be a discrete semigroup. The following statements are equivalent.

(a) $\beta S$ is right cancellative.
(b) $K(\beta S)$ contains an element which is right cancelable in $\beta S$.
(c) $\beta S$ has a right cancelable element and $K(\beta S) = \beta S$.
(d) Every idempotent in $\beta S$ is a right identity for $\beta S$.
(e) There exist a finite group $G \subseteq S$ and a compact Hausdorff left zero semigroup $X$ such that $\beta S = XG$ and the function $\varphi : X \times G \to \beta S$ defined by $\varphi(x, g) = xg$ is both an isomorphism and a homeomorphism.
(f) There exist a finite group $G$ and a left zero semigroup $L$ such that $S$ is isomorphic to $L \times G$.
(g) There exist a finite group $G$ and a discrete left zero semigroup $L$ such that $\beta S$ is isomorphic to $\beta L \times G$.
(h) $S$ has a right cancelable element and $K(\beta S) = \beta S$.

Proof. With one exception the proof proceeds by exact left-right switches of the proof of Theorem 5. That exception is the portion of the proof that (d) implies (e) wherein we concluded that $T$ was compact. (It is true that $X = \bigcap_{y \in \beta S} \lambda_y^{-1}\{y\}$, but we only know that $\lambda_y$ is continuous for $y \in S$.) Fortunately, (d) states that $X = E(\beta S)$ is the set of right identities of $\beta S$. We claim that $X$ is precisely the set of right identities
of $S$. Indeed, given $f \in \beta S$ such that $yf = y$ for all $y \in S$, one has that $\rho_f$ and the identity agree on $S$ and therefore on $\beta S$. Consequently $X = \bigcap_{y \in S} \lambda_y^{-1}[\{y\}]$, and so $X$ is compact. 

7 Corollary. If $S$ is a discrete semigroup and if $K(\beta S)$ contains an element left cancelable in $\beta S$ and an element right cancelable in $\beta S$, then $S$ is a finite group.

Proof. By either of the above theorems we have that $K(\beta S) = \beta S$ and each idempotent of $\beta S$ is a two sided identity for $\beta S$. Thus by Lemma 3 we have an idempotent $e \in S$ such that $G = e\beta S \cap \beta Se$ is a finite group. Since $e$ is both a left identity and a right identity for $S$ we have that $G = \beta S$. 

References


Neil Hindman 
Department of Mathematics 
Howard University 
Washington, DC 20059 
USA 
hhindman@aol.com

Dona Strauss 
Mathematics Centre 
University of Hull 
Hull HU6 7RX 
UK 
d.strauss@hull.ac.uk