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Algebra in the Stone-Čech Compactification and its Applications to Ramsey Theory

A printed lecture presented to the *International Meeting of Mathematical Sciences*

by

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Let me begin by expressing my sincere gratitude to the *Japanese Association of Mathematical Sciences* for inviting me to present this lecture and for giving me the *JAMS International Prize for 2003*. I am deeply honored.

This lecture is not a survey, but simply a discussion of some topics that I find interesting. For the most recent surveys of this subject area in which I have participated see [10] and [12] and, for a wealth of detail, see the book [11].

1. Some history.

Please forgive me for starting with a rather lengthy recital of personal history. (An advantage of a printed lecture is that it is easy for those in the audience who are bored to press an individualized fast forward button.) I was raised as a topologist and significant portions of my dissertation involved the remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification of the positive integers. In general, the points of the Stone-Čech compactification of a discrete space D can be taken to be the ultrafilters on D , with a point x of D identified with the principal ultrafilter $\{A \subseteq D : x \in A\}$. Because ultrafilters are the points of βD we customarily denote them by lower case letters.

An ultrafilter on a set D is a subset of $\mathcal{P}(D)$ which is maximal with respect to the finite intersection property – alternatively, it is a maximal filter on D . Ultrafilters can also be viewed as $\{0, 1\}$ -valued measures on $\mathcal{P}(D)$. Such a measure characterizes every subset of D as either “large” or “small”. Given an ultrafilter p as we have defined it and that same ultrafilter viewed as a measure μ and given a set $A \subseteq D$, the statements $A \in p$ and $\mu(A) = 1$ are synonymous.

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Sometime around 1971, Fred Galvin asked Paul Erdős whether there were any *almost translation invariant* ultrafilters on \mathbb{N} . That is, did there exist an ultrafilter p on \mathbb{N} such that, whenever $A \in p$, $\{x \in \mathbb{N} : -x + A \in p\} \in p$, where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$? The terminology can best be understood by viewing the ultrafilter as a measure. The measure μ is almost translation invariant if and only if, given any large set A , the translation $-x + A$ is μ -almost always large.

Soon thereafter Erdős encountered W. Wistar Comfort, an author of *The Theory of Ultrafilters* [6], and asked him whether almost translation invariant ultrafilters existed. Comfort, who had been my dissertation advisor, in turn relayed the question to me. I originally put the question on the shelf. But, hearing that Erdős would be visiting the Claremont Colleges (while I was employed at Cal State L.A.), I started thinking a little about the problem so I would have something to talk to him about. When I did see him I found out that the question originated with Galvin who was then at UCLA.

Unfortunately (or perhaps fortunately – I'll explain that in a minute) somewhere in the relays above, the definition of an almost translation invariant ultrafilter mutated slightly. That is, it changed to “whenever $A \in p$, $\{x \in \mathbb{N} : x + A \in p\} \in p$ ”. That is, as opposed to the original question about the existence of downward almost translation invariant ultrafilters, the question became one of the existence of upward almost translation invariant ultrafilters.

I was able to show quite easily that no such ultrafilter exists. (And this was the reason for the “perhaps fortunately” remark above; if I had addressed the original, much more difficult question, from the start, I might well have simply given up. As it was, by the time I found out about the original question I was hooked.) To see that there is no upward almost translation invariant ultrafilter, suppose that p is such an ultrafilter and for $i \in \{1, 2\}$ let $A_i = \{x \in \mathbb{N} : \text{the rightmost nonzero digit in the ternary expansion of } x \text{ is } i\}$. Then one of these A_i 's is a member of p . So $A_i \cap \{x \in \mathbb{N} : x + A_i \in p\} \in p$. Also, either there is some $k \in \omega = \mathbb{N} \cup \{0\}$ such that the set $3^k i + 3^{k+1} \omega$ of numbers whose least significant digit is in position k is a member of p or for every $k \in \omega$, $\{3^l i + 3^{l+1} a : l > k \text{ and } a \in \omega\}$ is in p . In the first case pick $x = 3^k i + 3^{k+1} a \in A_i \cap \{x \in \mathbb{N} : x + A_i \in p\}$, let $B = A_i \cap (x + A_i)$. Then $B \in p$ so $B \cap \{y \in \mathbb{N} : y + B \in p\} \in p$. So pick $y = 3^k i + 3^{k+1} b \in B \cap \{y \in \mathbb{N} : y + B \in p\}$ and pick $z = 3^k i + 3^{k+1} c \in B \cap (y + B)$. Then $z - y \in B \subseteq x + A_i$ while $z - y - x \notin A_i$. In the second case pick $x = 3^k i + 3^{k+1} a \in A_i \cap \{x \in \mathbb{N} : x + A_i \in p\}$ and pick $y = 3^l + 3^{l+1} b \in A_i \cap (x + A_i)$ with $l > k$. Then $y - x \notin A_i$.

I was quite excited with this proof, and phoned Galvin, who congratulated me and informed me that I had answered the wrong question. Somewhat deflated, I asked him

why he wanted to know about the existence of (downward) almost translation invariant ultrafilters. He informed me that if such things existed, they would provide a simple proof of a conjecture of Ronald Graham and Bruce Rothschild. This was that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$, there must exist some $i \in \{1, 2, \dots, r\}$ and some sequence $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$, where $FS(\langle x_n \rangle_{n=1}^\infty) = \{\sum_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\}$. Notice that $FS(\langle x_n \rangle_{n=1}^\infty)$ includes terms like $x_1 + x_3 + x_{12}$, but not $x_2 + x_2$.

Galvin's simple proof ran as follows. Let p be an almost translation invariant ultrafilter and let $\mathbb{N} = \bigcup_{i=1}^r A_i$. Then some $A_i \in p$. Let $B_1 = A_i$ and pick $x_1 \in B_1$ such that $-x_1 + B_1 \in p$. Let $B_2 = B_1 \cap (-x_1 + B_1)$. Inductively, given B_n , choose $x_n \in B_n$ such that $-x_n + B_n \in p$ and let $B_{n+1} = B_n \cap (-x_n + B_n)$. One then easily shows by induction on $|F|$ that if $k = \min F$, then $\sum_{n \in F} x_n \in B_k$. Less formally, let us see for example why $x_2 + x_7 + x_8 \in A_i$. One has $x_8 \in B_8 \subseteq (-x_7 + B_7)$ so $x_7 + x_8 \in B_7 \subseteq B_6 \subseteq \dots \subseteq B_3 \subseteq (-x_2 + B_2)$ and therefore $x_2 + x_7 + x_8 \in B_2 \subseteq B_1 = A_i$.

With some significant effort, I succeeded in showing that if one assumes the continuum hypothesis, then the Graham-Rothschild conjecture implied the existence of almost translation invariant ultrafilters. And, with a great deal more effort I established (with an elementary but very complicated proof) that the Graham-Rothschild conjecture is indeed valid. (I shall refer to that result henceforth as the Finite Sums Theorem.) If the reader has a graduate student that she wants to punish, she should make him read and understand that original proof in [9].

At any rate, the situation at the end of 1972 was that Galvin's almost translation invariant ultrafilters were figments of the continuum hypothesis. Galvin continued to want to know if they really existed, that is, whether their existence could be established in ZFC. In 1975 he encountered Steven Glazer and asked him whether such ultrafilters could be shown to exist. Glazer immediately answered "yes". Galvin tried to explain that he must not understand the question, because the answer couldn't be that easy. It turned out that it was!

Glazer knew three things that were relevant. The first of these was that any compact right topological semigroup has idempotents. This fact is due to Robert Ellis [7, Corollary 2.10]. (A semigroup (S, \cdot) which is also a topological space is *right topological* if and only if for every $x \in S$, the operation $\rho_x : S \rightarrow S$ defined by $\rho_x(y) = y \cdot x$ is continuous.) The second relevant fact was that $\beta\mathbb{N}$ has a natural operation extending addition on \mathbb{N} which makes $(\beta\mathbb{N}, +)$ a right topological semigroup.

A fair number of mathematicians knew both of these facts. More generally, given

any discrete semigroup (S, \cdot) the operation can be extended to the Stone-Čech compactification βS so that $(\beta S, \cdot)$ is a right topological semigroup with the additional property that λ_x is continuous for each $x \in S$, where $\lambda_x(y) = x \cdot y$. The third relevant fact was that $\beta \mathbb{N}$ is naturally viewed as a space of ultrafilters, and that given p and q in $\beta \mathbb{N}$ and $A \subseteq \mathbb{N}$, $A \in p + q$ if and only if $\{x \in \mathbb{N} : -x + A \in q\} \in p$. Knowing these things, the answer was indeed immediate. That is, an almost translation invariant ultrafilter is exactly an idempotent.

Since the proof of the existence of idempotents in a compact right topological semigroup is quite short, one thereby had a short proof of the Finite Sums Theorem. When Galvin wrote to me with this proof in the fall of 1975, I was very excited. At that time, I fell in love with the algebraic structure of the Stone-Čech compactification of a discrete semigroup, and have spent the rest of my mathematical career investigating this structure and its applications to Ramsey Theory – that part of combinatorics which finds regular structures in one cell of a partition of some given set, the Finite Sums Theorem being one such result.

In the remainder of this lecture, I shall discuss some of the applications of the algebra of βS to Ramsey Theory. In Section 2 I will discuss results whose first (and often only) proof was obtained using this algebraic structure as well as results whose proofs are vastly simplified by use of this structure. In the third and final section I shall discuss central sets. These are sets with very rich combinatorial structure that have not gotten as wide attention as I believe they deserve.

Let me conclude this historical section with a more recent tale and some advice that I have often given to young mathematicians. This is advice that has done wonders for me over the years. That is to find someone who is smarter than you are and get them to put your name on their papers.

Principal among such people is Dona Strauss, with whom I wrote [11] and have collaborated on numerous other papers. While we were in the process of writing that book, Dona came up with a new, and even shorter, proof of the Finite Sums Theorem. Given an idempotent $p \in \beta \mathbb{N}$ and $A \in p$, she defined $A^* = \{x \in A : -x + A \in p\}$. She then wrote that, for any $x \in A^*$, $-x + A^* \in p$. I tried to explain to her that A^* is significantly smaller than A and all one knew was that $-x + A \in p$. It turned out that she was right. (Given $x \in A^*$, let $B = -x + A$. Then $B^* \in p$ and $B^* \subseteq -x + A^*$.) Using this fact one proves the Finite Sums Theorem by taking an idempotent p and some $A \in p$ and inductively choosing $\langle x_n \rangle_{n=1}^\infty$ so that for each $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=1}^m) \subseteq A^*$. Given $\langle x_n \rangle_{n=1}^m$, choose $x_{m+1} \in A^* \cap \bigcap \{-y + A^* : y \in FS(\langle x_n \rangle_{n=1}^m)\}$.

2. Anything you can do, we can do better.

Shortly after learning of the Galvin-Glazer proof of the Finite Sums Theorem, I obtained a new result quite easily. It was known to be an easy consequence of the Finite Sums Theorem itself that if any semigroup (S, \cdot) is partitioned into finitely many classes, then one of these classes contains $FP(\langle x_n \rangle_{n=1}^\infty)$ for some sequence $\langle x_n \rangle_{n=1}^\infty$, where $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\}$ and the products are taken in increasing order of indices. In particular, if $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$, then there exist i and j and sequences $\langle x_n \rangle_{n=1}^\infty$ and $\langle y_n \rangle_{n=1}^\infty$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$ and $FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A_j$. But one did not know whether one could choose $i = j$.

It was easy to show that $cl_{\beta\mathbb{N}}\{p \in \beta\mathbb{N} : p + p = p\}$ is a subsemigroup of $(\beta\mathbb{N}, \cdot)$. Since $q \in cl_{\beta\mathbb{N}}\{p \in \beta\mathbb{N} : p + p = p\}$ if and only if each $A \in q$ contains $FS(\langle x_n \rangle_{n=1}^\infty)$ for some sequence $\langle x_n \rangle_{n=1}^\infty$, one thus had a proof that one can indeed choose $i = j$. (We know, however, that one cannot also choose $\langle x_n \rangle_{n=1}^\infty = \langle y_n \rangle_{n=1}^\infty$.) It was more than 15 years later that an elementary proof of this fact was found. This was done in collaboration with Vitaly Bergelson [3], another of those mathematicians that I referred to at the end of the first section. If the reader wants more details of results mentioned here that were obtained before 1998, they can be located by way of the book [11].

Consider the Finite Sums Theorem (whose original proof was elementary and combinatorial in nature). From this theorem itself comes the following superficial strengthening. (This theorem can be stated more precisely using the notion of a *tree*. See for example [11].) When we say that a set is “finitely colored” we mean that there is a function from that set to a finite set.

Theorem. *Let \mathbb{N} be finitely colored. There is one color and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty)$ contained in the specified color class. Moreover, having chosen $\langle x_n \rangle_{n=1}^m$, one has infinitely many choices for x_{m+1} .*

This is a trivial consequence of the Finite Sums Theorem, because if $FS(\langle x_n \rangle_{n=1}^\infty)$ is monochrome, then x_m could be replaced by any x_k for $k > m$. However, the following is a genuine strengthening which does not have any known elementary proofs.

Theorem. *Let \mathbb{N} be finitely colored. There is one color and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty)$ contained in the specified color class. Moreover, having chosen $\langle x_n \rangle_{n=1}^m$, there is a set with positive upper density, any member of which can be chosen as x_{m+1} .*

This is an easy consequence of either of the algebraic proofs given in the first section. The set $\Delta = \{p \in \beta\mathbb{N} : \text{every member of } p \text{ has positive upper density}\}$ is a compact

subsemigroup of $\beta\mathbb{N}$ (in fact a left ideal of $\beta\mathbb{N}$). So there exist idempotents in Δ and starting with any such idempotent yields the result above.

Two old Ramsey Theoretic results are van der Waerden's Theorem and the Hales-Jewett Theorem. The first of these says that whenever \mathbb{N} is partitioned into finitely many cells, one cell contains arbitrarily long arithmetic progressions. To state the Hales-Jewett Theorem, we need to introduce some terminology. Let A be a finite alphabet, let S_0 be the set of words over A , and let S_1 be the set of words over $A \cup \{v\}$ in which v occurs, where v is a “variable” not in A . Given a “variable word” $w \in S_1$ and given $a \in A$, $w\langle a \rangle$ is the result of replacing each occurrence of v by a . For example, if $A = \{a, b, c\}$ and $w = avabbca$ then $w\langle b \rangle = ababbca$. The Hales-Jewett theorem says that if $r \in \mathbb{N}$ and $S_0 = \bigcup_{i=1}^r C_i$, then there exist $i \in \{1, 2, \dots, r\}$ and $w \in S_1$ such that $\{w\langle a \rangle : a \in A\} \subseteq C_i$.

Hillel Furstenberg and Yitzhak Katznelson in the late 1980's came up with a proof of van der Waerden's Theorem using the algebraic structure of an enveloping semigroup. Bergelson noticed that their proof could be simplified by using the algebraic structure of $\beta\mathbb{N}$ and the four of us published the proof in that context [2]. Andreas Blass noticed that essentially the same proof established the Hales-Jewett Theorem, and he, Bergelson, and I published several extensions of that theorem [1].

A much more recent development is the algebraic proof of the Graham-Rothschild Parameter Sets Theorem. In order to discuss this result, I need to introduce some more terminology.

Throughout the rest of this section A will denote a finite nonempty alphabet. We choose a set $V = \{v_n : n \in \omega\}$ (of *variables*) such that $A \cap V = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup V$, with concatenation as the semigroup operation. (Formally a *word* w is a function from an initial segment $\{0, 1, \dots, k-1\}$ of ω to the alphabet and the length $\ell(w)$ of w is k . We shall occasionally need to resort to this formal meaning, so that if $i \in \{0, 1, \dots, \ell(w)-1\}$, then $w(i)$ denotes the $(i+1)^{\text{st}}$ letter of w .)

For each $n \in \mathbb{N}$, we define W_n to be the set of words over the alphabet $A \cup \{v_0, v_1, \dots, v_{n-1}\}$ and we define W_0 to be the set of words over A . We note that each W_n is a subsemigroup of W .

Let $n \in \mathbb{N}$, let $k \in \omega$ with $k \leq n$, and let $\emptyset \neq B \subseteq A$. Then $[B]_k^{(n)}$ is the set of all words w over the alphabet $B \cup \{v_0, v_1, \dots, v_{k-1}\}$ of length n such that

- (1) for each $i \in \{0, 1, \dots, k-1\}$, if any, v_i occurs in w and
- (2) for each $i \in \{0, 1, \dots, k-2\}$, if any, the first occurrence of v_i in w precedes the first

occurrence of v_{i+1} .

Let $k \in \mathbb{N}$. Then the set of k -variable words is $S_k = \bigcup_{n=k}^{\infty} [A]^{(n)}$. Also $S_0 = W_0$.

Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u \rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \dots, \ell(w) - 1\}$

$$w\langle u \rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ u(j) & \text{if } w(i) = v_j. \end{cases}$$

That is, $w\langle u \rangle$ is the result of substituting $u(j)$ for each occurrence of v_j in w .

The following theorem is commonly known as the Graham-Rothschild Theorem.

Theorem (Graham-Rothschild). *Let $m, n \in \omega$ with $m < n$, and let S_m be finitely colored. There exists $w \in S_n$ such that $\{w\langle u \rangle : u \in [A]^{(n)}\}$ is monochrome.*

To discuss our algebraic extension of the Graham-Rothschild Theorem, as well as set the stage for our discussion of central sets in the next section, we need to introduce the notion of minimal idempotents. In any semigroup with idempotents, there is a partial ordering of the idempotents defined by $p \leq q$ if and only if $p = pq = qp$. In any compact right topological semigroup T , there is a smallest two sided ideal $K(T)$. This ideal has a rich structure which we will not discuss here. The important fact for us now is that an idempotent is minimal with respect to the ordering \leq defined above if and only if it is a member of the smallest ideal.

Now, given $n \in \mathbb{N}$ and $u \in [A]^{(n-1)}$, define $h_u : S_n \rightarrow S_{n-1}$ by $h_u(w) = w\langle u \rangle$. Denote by \widetilde{h}_u the continuous extension of h_u which takes βS_n to βS_{n-1} . The following result was proved in [5].

Theorem. *Let p be a minimal idempotent in βS_0 . There is a sequence $\langle p_n \rangle_{n=0}^{\infty}$ such that*

- (1) $p_0 = p$;
- (2) for each $n \in \mathbb{N}$, p_n is a minimal idempotent of βS_n ;
- (3) for each $n \in \mathbb{N}$, $p_n \leq p_{n-1}$;
- (4) for each $n \in \mathbb{N}$ and each $u \in [A]^{(n-1)}$, $\widetilde{h}_u(p_n) = p_{n-1}$.

The following extension of the Graham-Rothschild Theorem is an easy corollary. (It is a not very well known fact that this extension is also a consequence of results in [4], results that were also obtained by using algebra in the Stone-Čech compactification.)

Corollary. *Assume that for each $n \in \omega$, S_n has been finitely colored. Then, there exists*

a sequence $\langle w_n \rangle_{n < \omega}$ with each $w_n \in S_n$ such that for every $m \in \omega$,

$$S_m \cap \left\{ \prod_{n \in F} w_n \langle u_n \rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F, u_n \in \bigcup_{i=0}^{\min F} [A] \binom{n}{i} \right\}$$

is monochrome. (That is, the color of $\prod_{n \in F} w_n \langle u_n \rangle$ is determined solely by the number of variables in $\prod_{n \in F} w_n \langle u_n \rangle$.)

3. The centrality of central sets.

In [8] Furstenberg introduced the notion of *central* subsets of \mathbb{N} . His definition was in terms of the notions of uniform recurrence and proximality from topological dynamics. He showed that whenever \mathbb{N} is finitely colored, one color class must be central (so that any results about central sets are automatically results about some color class of any finite coloring). He then proved the following theorem.

Theorem (Central Sets Theorem for \mathbb{N}). *Let C be a central subset of \mathbb{N} , let $k \in \mathbb{N}$, and for each $t \in \{1, 2, \dots, k\}$, let $\langle y_{t,n} \rangle_{n=1}^\infty$ be a sequence in \mathbb{N} . There exist a sequence $\langle a_n \rangle_{n=1}^\infty$ in \mathbb{N} and a sequence $\langle H_n \rangle_{n=1}^\infty$ of finite nonempty subsets of \mathbb{N} such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f : \mathbb{N} \rightarrow \{1, 2, \dots, k\}$*

$$FS(\langle a_n + \sum_{t \in H_n} y_{f(n),t} \rangle_{n=1}^\infty) \subseteq C.$$

This theorem may seem a bit obscure at first glance. But it has many significant consequences. For example, any central subset of \mathbb{N} contains arbitrarily long arithmetic progressions (and so the Central Sets Theorem implies van der Waerden's Theorem). In fact, one can choose the increment out of the set of finite sums of any prespecified sequence. (To see this, let $\langle x_n \rangle_{n=1}^\infty$ and $k \in \mathbb{N}$ be given. For $t \in \{1, 2, \dots, k\}$ and $n \in \mathbb{N}$, let $y_{t,n} = tx_n$. Pick $\langle a_n \rangle_{n=1}^\infty$ and $\langle H_n \rangle_{n=1}^\infty$ as guaranteed by the Central Sets Theorem. Then $\{a_1 + \sum_{n \in H_1} y_{1,n}, a_1 + \sum_{n \in H_1} y_{2,n}, \dots, a_1 + \sum_{n \in H_1} y_{k,n}\}$ is a length k arithmetic progression with increment $\sum_{n \in H_1} x_n$.)

More generally, if $m, n \in \mathbb{N}$ and B is any $m \times n$ kernel partition regular matrix (meaning that whenever \mathbb{N} is finitely colored there is a monochrome $\vec{x} \in \mathbb{N}^n$ such that $B\vec{x} = \vec{0}$), then for any central set C in \mathbb{N} there will exist $\vec{x} \in C^n$ such that $B\vec{x} = \vec{0}$.

Bergelson then had the idea that perhaps we could prove the same theorem for members of minimal idempotents in $\beta\mathbb{N}$. It turned out that he was correct. Moreover, with the help of Benjamin Weiss we were able to show that a subset of \mathbb{N} is central if and only if it is a member of a minimal idempotent. Even though I had a hand in this proof, I have not the foggiest notion of how the idea came to Bergelson. It is not at all obvious that these notions would be the same. It is not even obvious that the

notion of central as defined by Furstenberg is closed under passage to supersets, while it is trivial that any superset of a member of a minimal idempotent is a member of that same idempotent.

Moreover, the algebraic characterization makes sense in any semigroup. And a nearly verbatim version of the Central Sets Theorem is valid in any commutative semigroup. (The Central Sets Theorem for noncommutative semigroups is much more complicated to state.)

In Section 2 we saw that either of the algebraic proofs given in the first section showed that in constructing a sequence with monochrome finite sums, one could make the choice of each term from a set with positive upper density. We have a more impressive result in terms of central sets.

Theorem. *Let \mathbb{N} be finitely colored. There is one color and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty)$ contained in the specified color class. Moreover, having chosen $\langle x_n \rangle_{n=1}^m$, one can choose x_{m+1} as any member of some specified central set.*

This is also an easy consequence of either of the algebraic proofs we described in the first section. Just start with a minimal idempotent. Neither of the extensions that we have mentioned are consequences of the Finite Sums Theorem, because one can easily prevent $FS(\langle x_n \rangle_{n=1}^\infty)$ from being central or having positive upper density. For example, one can color the interval $\{2^n, 2^n + 1, \dots, 2^{n+1} - 1\}$ one color according to whether n is even or odd. (Then if $FS(\langle x_n \rangle_{n=1}^\infty)$ is monochrome, the sequence $\langle x_n \rangle_{n=1}^\infty$ can choose at most one member from any interval $\{2^m, 2^m + 1, \dots, 2^{m+1} - 1\}$ and thus will not contain a length three arithmetic progression, so $\{x_n : n \in \mathbb{N}\}$ certainly can not be central and clearly does not have positive upper density.)

I believe that it is very unlikely that an elementary proof of the above theorem will be obtained anytime soon, meaning within the lifetime of my children – either biological or mathematical.

We know of only one naturally stated coloring theorem about \mathbb{N} that is not satisfied by every central subset of \mathbb{N} . This is the fact that for any finite coloring, one color class must have positive upper density. The set $\beta\mathbb{N} \setminus (\mathbb{N} \cup \Delta)$ of nonprincipal ultrafilters that have a member with zero density is a left ideal of $\beta\mathbb{N}$ so contains a minimal idempotent.

Let me conclude by describing the only other Ramsey Theoretic result in *any* semigroup that I know of for which central sets are not good enough. That is the second simplest instance of the Graham-Rothschild Theorem. (The simplest instance is the Hales-Jewett Theorem, and any central subset of S_0 satisfies the conclusion of the

Hales-Jewett Theorem. (The following is taken from [5].)

Theorem. *There is a central subset M of S_1 such that there is no $w \in S_2$ with the property that $w\langle u \rangle \in M$ for every $u \in [A]^{(2)}_1$.*

Proof. Recall that we are assuming that $A \neq \emptyset$, so pick $a \in A$. For each $k \in \mathbb{N}$, let $L_k = \{w \in S_1 : |\{i : w(i) = v_0\}| \geq |\{i : w(i) = a\}| + k\}$ and let $L = \bigcap_{k=1}^{\infty} \overline{L_k}$. Trivially $L \neq \emptyset$. Given any $w \in S_1$, if m is the length of w then for each $z \in L_{m+k}$, one has $wz \in L_k$. Consequently L is a left ideal of βS_1 . Pick a minimal idempotent $p \in L$. Let $M = L_1$. Then $M \in p$ so M is central.

Now let $w \in S_2$ and suppose that $w\langle u \rangle \in M$ for every $u \in [A]^{(2)}_1$. Let $u_1 = av_0$ and let $u_2 = v_0a$. Let

$$\begin{aligned} b &= |\{i : w(i) = a\}|, \\ c &= |\{i : w(i) = v_0\}|, \text{ and} \\ d &= |\{i : w(i) = v_1\}|. \end{aligned}$$

Then $d = |\{i : w\langle u_1 \rangle(i) = v_0\}| \geq |\{i : w\langle u_1 \rangle(i) = a\}| + 1 = b + c + 1$ and $c = |\{i : w\langle u_2 \rangle(i) = v_0\}| \geq |\{i : w\langle u_2 \rangle(i) = a\}| + 1 = b + d + 1$ and so $d \geq 2b + d + 2$, a contradiction. \square

My explanation of the above phenomenon is that it is an indication of exactly how strong the Graham-Rothschild Theorem is.

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¹ This paper is currently available at <http://members.aol.com/nhindman/>.

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